Self-consistent neutral current filaments in a relativistic collisionless plasma

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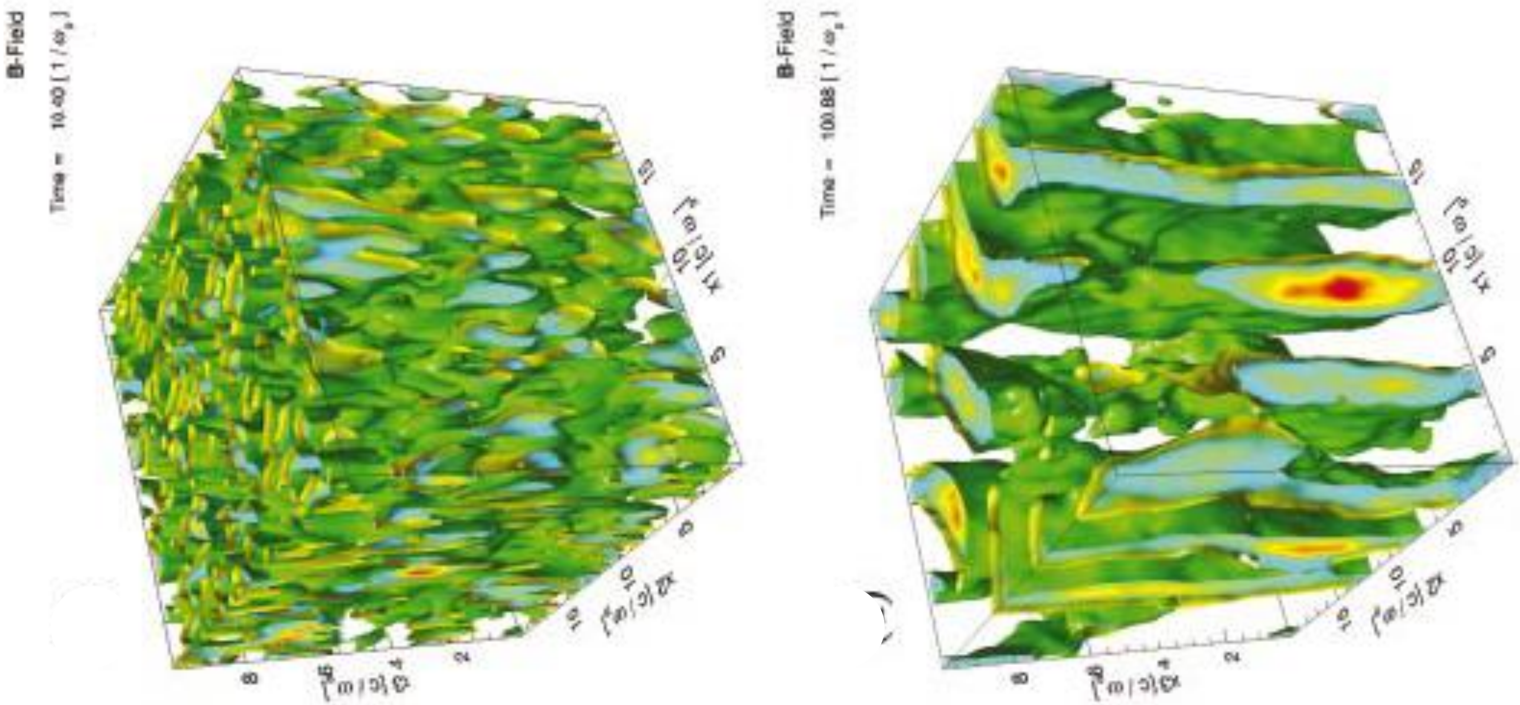
Abstract

An analytical description of a new broad class of the cylindrically symmetric magnetostatic structures (filaments) in a collisionless relativistic multicomponent plasma is given by means of the method of invariants of particle motion (PRL 104 (2010) 215002; Physics-Uspekhi 59 (2016) 1165).

Namely, the Vlasov-Maxwell equations are reduced to a nonlinear damped oscillator equation, with an explicit form of an effective nonlinear potential, for the particle distribution functions (PDFs) with a polynomial-exponential dependence on a longitudinal momentum. The particle distributions may be essentially different from the Maxwellian ones and vary along the radial direction self-consistently with the profile of azimuthal magnetic field of the filament.

We present a classification of possible radial profiles of a current density in these filaments, including zero, finite and infinite values of the total current as well as various spatial scales, both thin and thick with respect to a typical particle gyroradius.

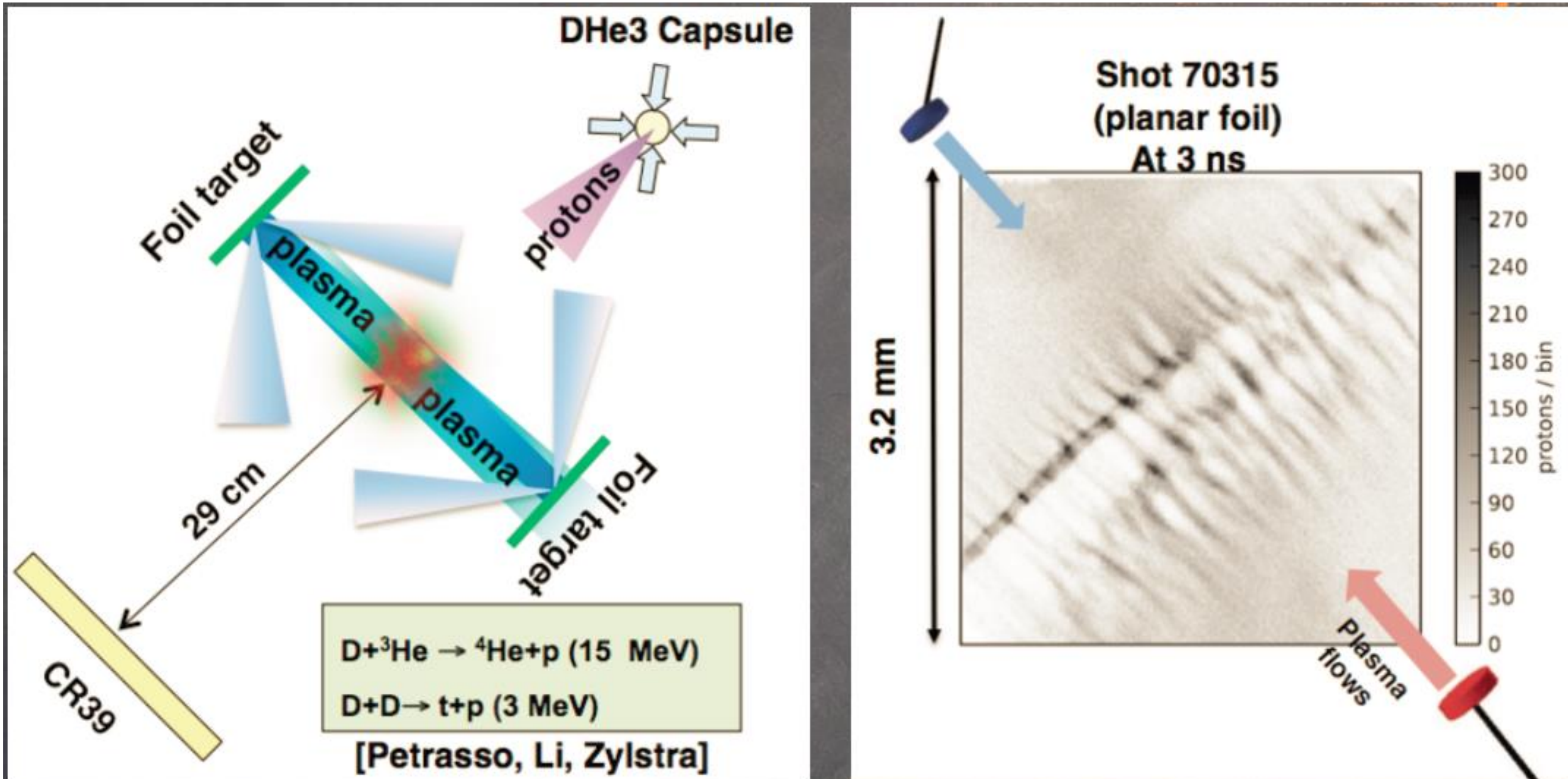
3D Weibel instability in e^- -e+ plasma



Magnetic field energy density for values of 15% of the maximum energy density. Results are shown slightly before saturation and in the quasi-static stage (ε_B~ 1%).

Fonseca, Silva et al (2003).

Collision of laser collisionless plasma jets



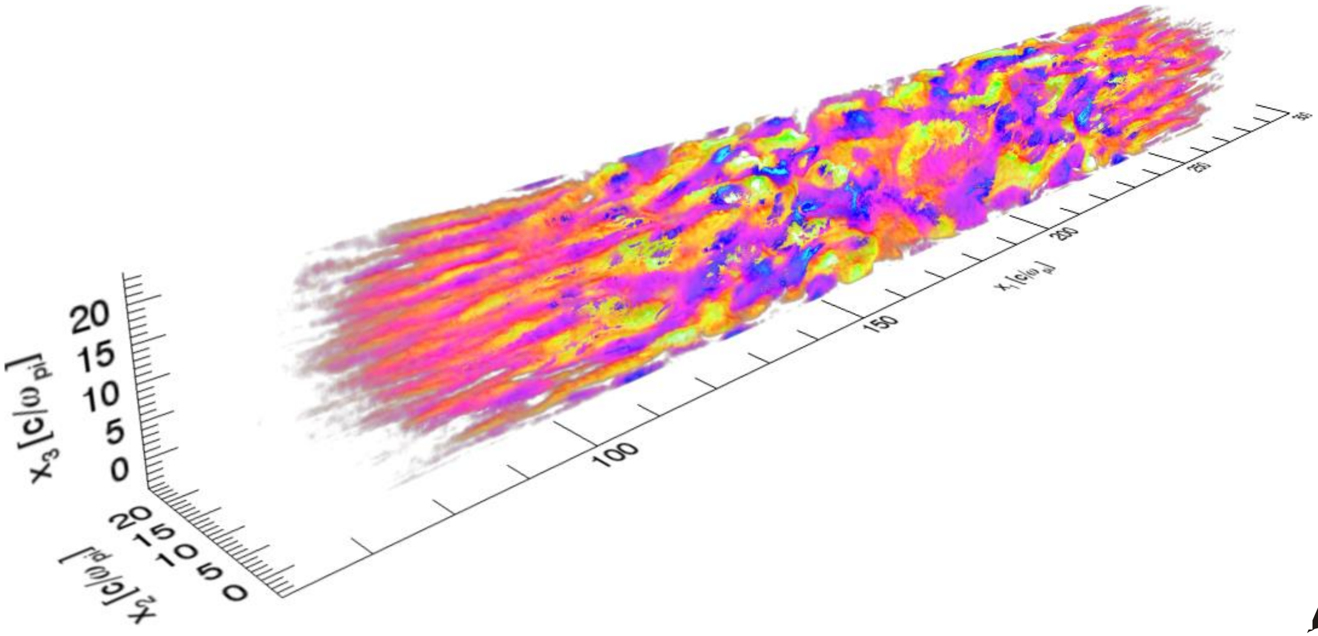
- First clearly seen filamentation from DHe3 capsule implosion protons. Later confirmed that same filamentation is seen with EP protons on Omega

Nonlinear evolution

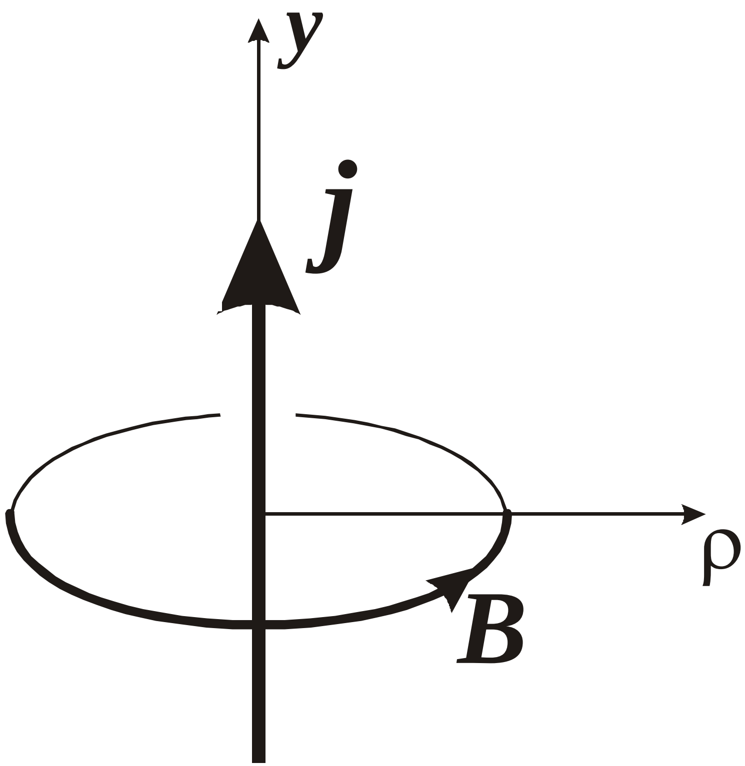
- Quasineutrality
- Magnetic energy can approach equipartition
- Current filaments merge due to Ampère force
- Spatial scale increases
- Slow magnetic field decay
- Metastable configurations

Equal treatment of relativistic and non-relativistic plasma

$$\langle B^2/8\pi \rangle \lesssim \langle (\gamma - 1) N m c^2 \rangle$$



Introduction



An example of magnetic field generation in a collisionless shock modelling.
Frederico Fiuza, Lawrence Livermore National Laboratory
(https://www.alcf.anl.gov/projects/particle-acceleration-shocks-astrophysics-laboratory-silico and http://arxiv.org/pdf/1210.2643.pdf).

Current density **j** and magnetic field **B** depend on radius

Examples of solutions with fixed energy distribution of particles: Bennett, 1934; Harris, 1962; Vasko, 2013.

For Bennett pinch with maxwellian PDF the momentum distribution is universal in space:

$$f_\alpha(\mathbf{r},\mathbf{p}) = N_\alpha(x)\tilde{f}_\alpha(\mathbf{p})$$

Maxwell-Vlasov equations

$$\mathbf{p}\frac{\partial f_\alpha}{\partial \mathbf{r}} + \frac{e_\alpha}{c}[\mathbf{p} \times [\nabla \times \mathbf{A}]]\frac{\partial f_\alpha}{\partial \mathbf{p}} = 0$$

$$[\nabla \times [\nabla \times \mathbf{A}]] = \frac{4\pi}{c}\sum_\alpha e_\alpha \int f_\alpha \frac{\mathbf{p}}{m_\alpha \gamma_a} d^3\mathbf{p}$$

Can there be cylindrical magnetostatic analytic solutions having arbitrary dependence on particle energy?

V.Ju. Martyanov, Vl.V. Kocharovsky, V.V. Kocharovsky, JETP **107**, 1049 (2008); Radiophys. Quant. Electr. **52**, n. 2, p. 79 (2009); Astronomy Lett. **36**, 396 (2010); Phys. Rev. Lett. **104**, 215002 (2010).

Assuming magnetic field is stronger than electric field, we work in that frame of reference where electric field vanishes (Gedalin et al., Journal of plasma physics **77**, n.2 (2011)):

$$\sum_\alpha e_\alpha \int f_\alpha \; d^3\mathbf{p} = 0$$

Integrals of particle motion

$$\mathcal{E} = c\sqrt{m^2c^2 + p^2} \qquad\qquad P_y = p_y + \frac{e}{c}A_y$$

Any PDF expressed as a function of integrals of particle motion satisfies the Vlasov equation (for a given sort of particles):

$$f = f(\mathcal{E}, P_y)$$

Distribution functions must be nonnegative and give finite particle density:

$$f \geq 0$$

$$\int f d^3\mathbf{p} < \infty$$

Grad-Shafranov equation and polynomial–exponential PDF

$$f_\alpha(\mathcal{E}, P_y) = \exp\left(\zeta_\alpha \frac{P_y}{m_\alpha c}\right) \sum_{i=0}^{d} f_{\alpha i}(\mathcal{E}) \left(\frac{P_y}{m_\alpha c}\right)^i$$

$$\frac{d^2 A}{d\rho^2} + \frac{1}{\rho}\frac{dA}{d\rho} = -\frac{dU(A)}{dA}$$

$$U = \sum_\alpha \exp\left(\frac{\zeta_\alpha e_\alpha A}{m_\alpha c^2}\right) \sum_{j=0}^{d} A^j \left\{ \sum_{i=j}^{d} \int f_{\alpha i}(\mathcal{E}) \left[Y_{\alpha ij}(p) - Y_{\alpha ij}(-p)\right] dp \right\}$$

$$Y_{\alpha ij}(p) = \exp\left(\frac{\zeta_\alpha p}{m_\alpha c}\right) \frac{4\pi^2 e_\alpha^j p \left(-\zeta_\alpha\right)^{j-i-3} i!}{\gamma_\alpha m_\alpha^{j-2} c^{2j-3} j!(i-j)!}.$$

$$\cdot \left(\exp(q)\Gamma(i-j+1, q)\left[(i-j+2)(i-j+1) - q^2\right] + q^{i-j+2} + (i-j+2)q^{i-j+1}\right)$$

$q = -\zeta_\alpha p / m_\alpha c$ $\quad$ $\exp(q)\Gamma(i-j+1, q)$ — a polynomial of order $i-j$

1D harmonic solution of the nonlinear problem (d=2)

$$\Delta_\perp A \; + k^2 A \; = 0$$

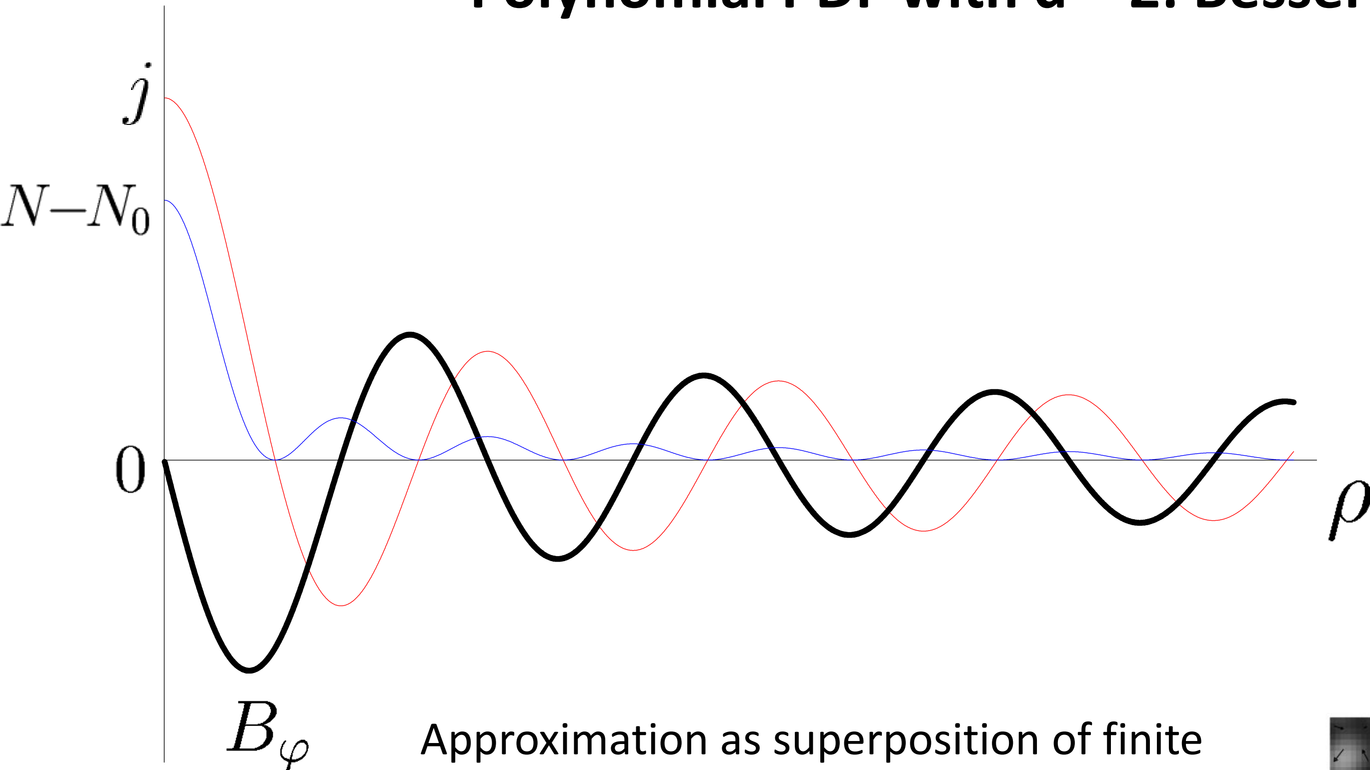
$$k^2 = \frac{32\pi^2}{3} \int f_2(\mathcal{E}) \frac{e^2 p^4}{m^3 c^4 \gamma} dp$$

$$A \; = A_{\max} \cos(kx)$$

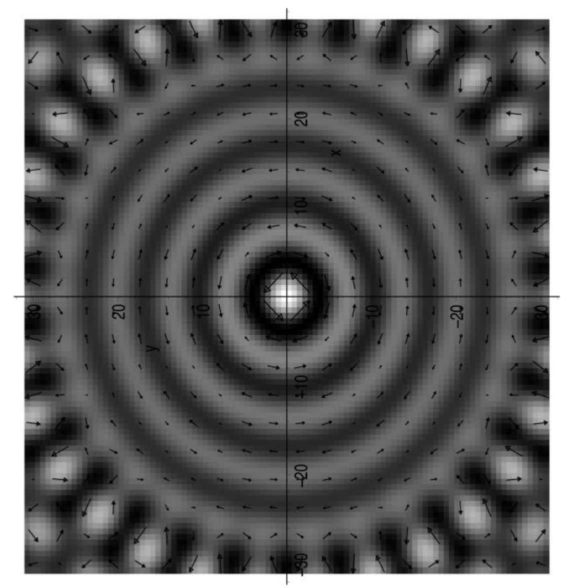
$$\frac{\langle W_B \rangle}{\langle W_e \rangle} = \frac{1}{3} \frac{\int f_2(\mathcal{E})(v^2/c^2)\gamma p^2 dp}{\int f_2(\mathcal{E})\gamma p^2 dp + \frac{2}{3}\int f_2(\mathcal{E}) p^2 c^2 \gamma p^2 \; dp / e^2 A_{\max}^2}$$

$$\frac{\langle W_B \rangle}{\langle W_e \rangle} = \frac{1}{3} \cdot \frac{1}{1 + \frac{2}{3}\frac{p^2 c^2}{e^2 A_{\max}^2}} \cdot \frac{v^2}{c^2} \qquad < \frac{1}{3}$$

Polynomial PDF with d = 2: Bessel solution



Approximation as superposition of finite number of planar harmonic perturbations:



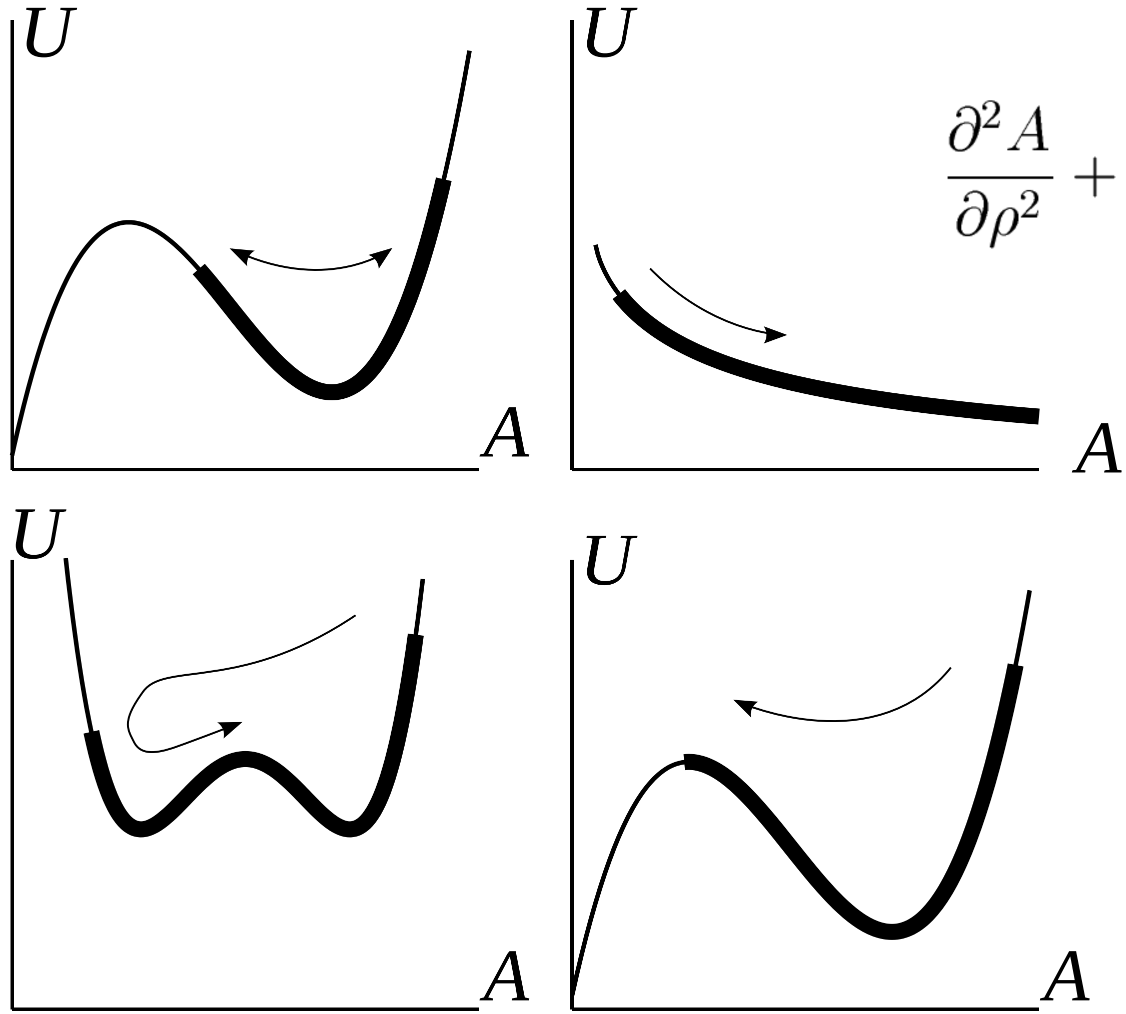
Stability condition is satisfied only up to a certain radius where the plasma is magnetized enough. Beyond this radius, particular properties of how the current filament is embedded in the background plasma should be considered.

Grad-Shafranov potential and effective viscous damping

Oscillating filament

Single filament

$$\frac{\partial^2 A}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial A}{\partial \rho} = -\frac{\partial U}{\partial A}$$



Complex filament

Shielded filament

Classification of solutions

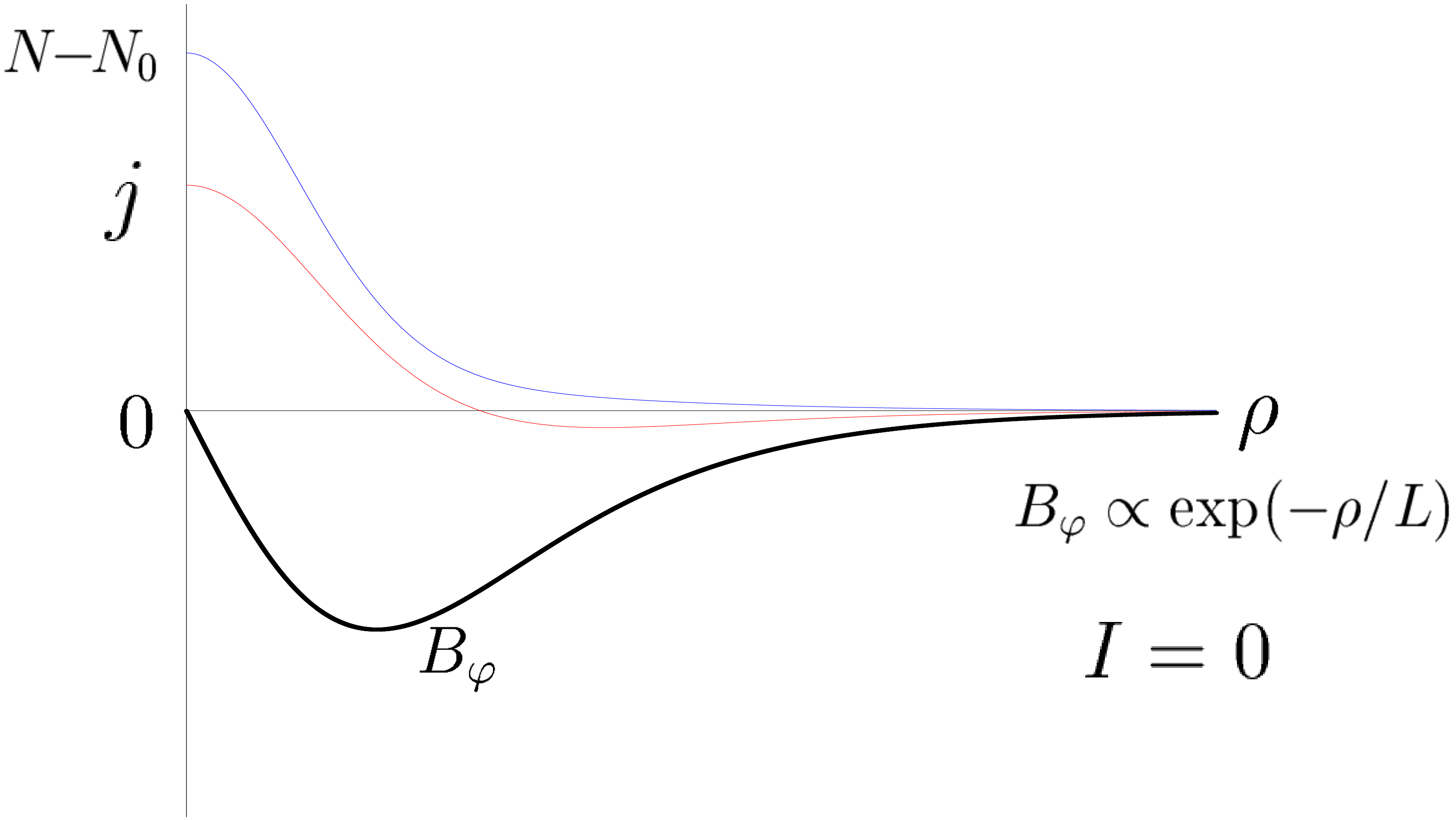
Vector potential A as a function of cylindrical radius can have

I) an infinite number of oscillations near a local minimum of potential U(A) (Bessel-like solution);

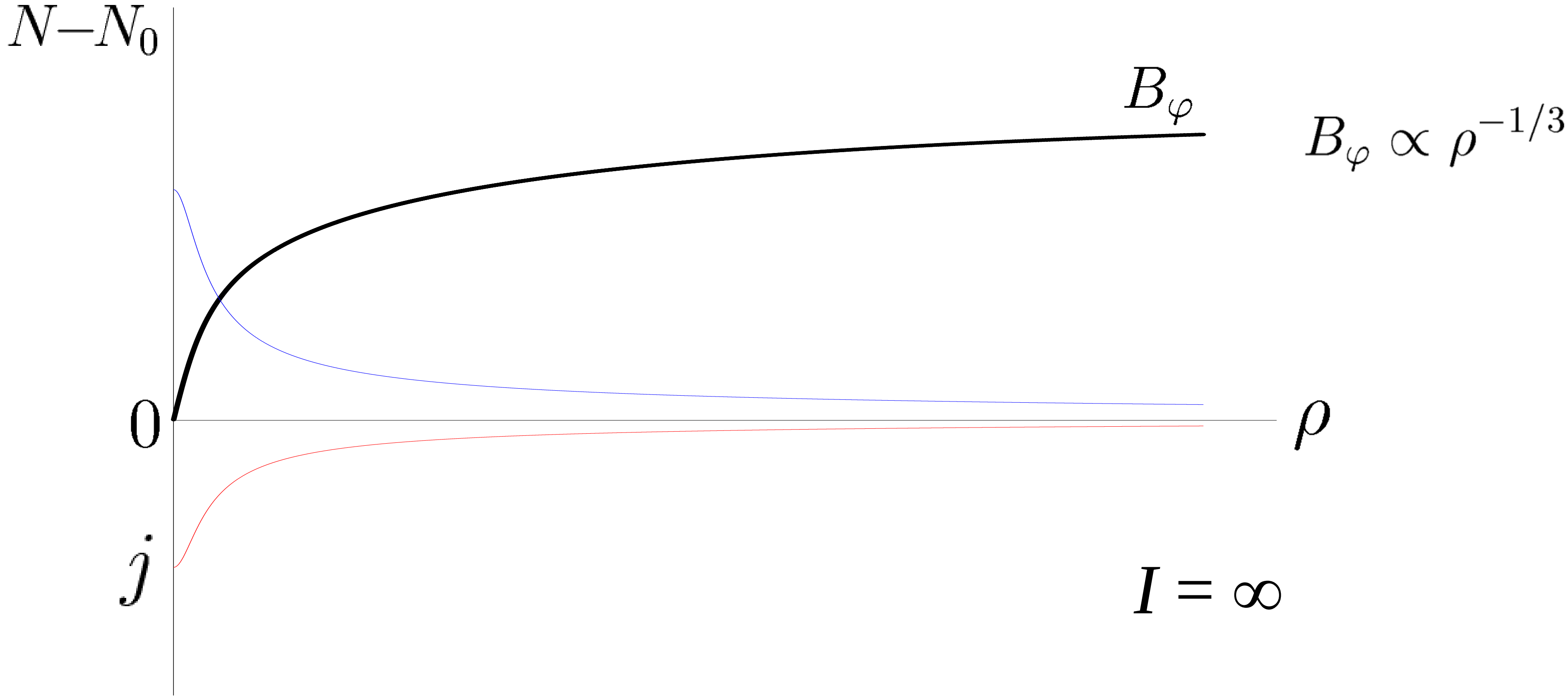
II) no oscillations and goes to infinity, admitting nonzero filament's net current, as in Bennett pinch. Current density can have the same sign throughout the filament and the particle densities can vanish at infinity;

III) a finite number of oscillations (or no oscillations) and stops at a local maximum of the Grad-Shafranov potential, meaning the filament's net current is zero.

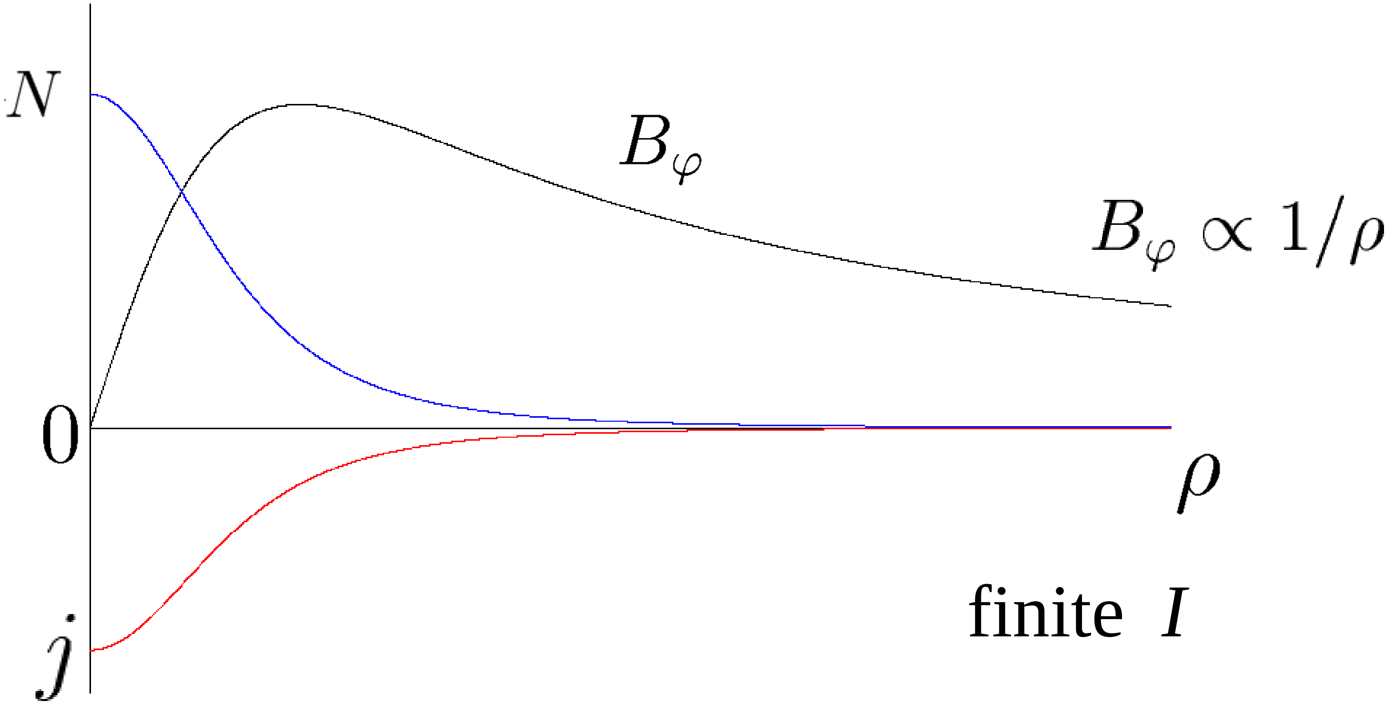
Shielded current filament (Taylor order d=3)



Single current filament (Taylor order d = -1)



Bennett pinch (PDF of exponential type)



$$A = -2A_0 \ln\left[1 + \frac{\alpha(x^2+y^2)}{8A_0}\right], \quad B = \frac{4A_0\alpha\rho}{8A_0 + \alpha\rho^2}, \quad N_j = N_{j\max}\left[1 + \frac{\alpha(x^2+y^2)}{8A_0}\right]^{-2}$$

Bennett pinch

The first example covers the *analytical generalization of the known Bennett pinch to the case of arbitrary energy distributions*, which corresponds to the exponential dependence of PDFs on the generalized momentum: $f_\alpha = F_{\alpha 0}(p) \exp(\zeta_\alpha P_z / m_\alpha c)$. In this case, the Grad–Shafranov equation takes the form

$$\frac{\mathrm{d}^2 A_z}{\mathrm{d}\rho^2} + \frac{1}{\rho} \frac{\mathrm{d}A_z}{\mathrm{d}\rho} = -\frac{W_0}{A_0} \exp\left(\frac{A_z}{A_0}\right), \tag{71}$$

where $A_0 = m_\alpha c^2 / \zeta_\alpha e_\alpha$ (assuming this quantity to be the same for all kinds of current-carrying particles α), while W_0 is positive and expressed as an integral including an arbitrary function $F_{\alpha 0}(p)$. The family of solutions satisfying the boundary condition $\mathrm{d}A_z/\mathrm{d}\rho|_{\rho\to 0} = 0$ is described by the formula (cf. Refs [1, 179])

$$A_z = -2A_0 \ln \frac{\sqrt{2W_0}(1+\kappa^2\rho^2)}{4\kappa A_0} \tag{72}$$

and is parametrized by a positive quantity κ, the inverse of the filament radius. If, for definiteness, $A_0 > 0$, κ is positive, too.

The corresponding magnetic field

$$B_\varphi = \frac{4A_0\kappa^2\rho}{1+\kappa^2\rho^2}, \tag{73}$$

unrelated to the energy distribution of particles, $F_{\alpha 0}$, is maximum at $\rho = \kappa^{-1}$, and amounts to $2\kappa A_0$. The current density is given by the expression (Fig. 3)

$$j_z = \frac{2cA_0\kappa^2}{\pi(1+\kappa^2\rho^2)^2}; \tag{74}$$

its simple integration gives a total current value of $2cA_0$ independent of κ and, therefore, of the filament radius.

Owing to the exponential dependence of the PDF on the projection of the generalized momentum, distributions of any kind of particles over momentum at all points of space are similar and differ only in the momentum-independent factor. Therefore, the shape of the particle concentration profile coincides with that of the current density profile: $N_\alpha = N_{\alpha\,\max}/(1+\kappa^2\rho^2)^2$. The integral of N_α over plane xy gives the number of particles per unit pinch length and is equal to $\pi\kappa^{-2}N_{\alpha\,\max}$, which once again confirms the possibility of conventionally regarding quantity κ^{-1} as the radius of the current filament being considered. This radius can be either longer or shorter than the particles' characteristic gyroradius depending on their energy distribution.

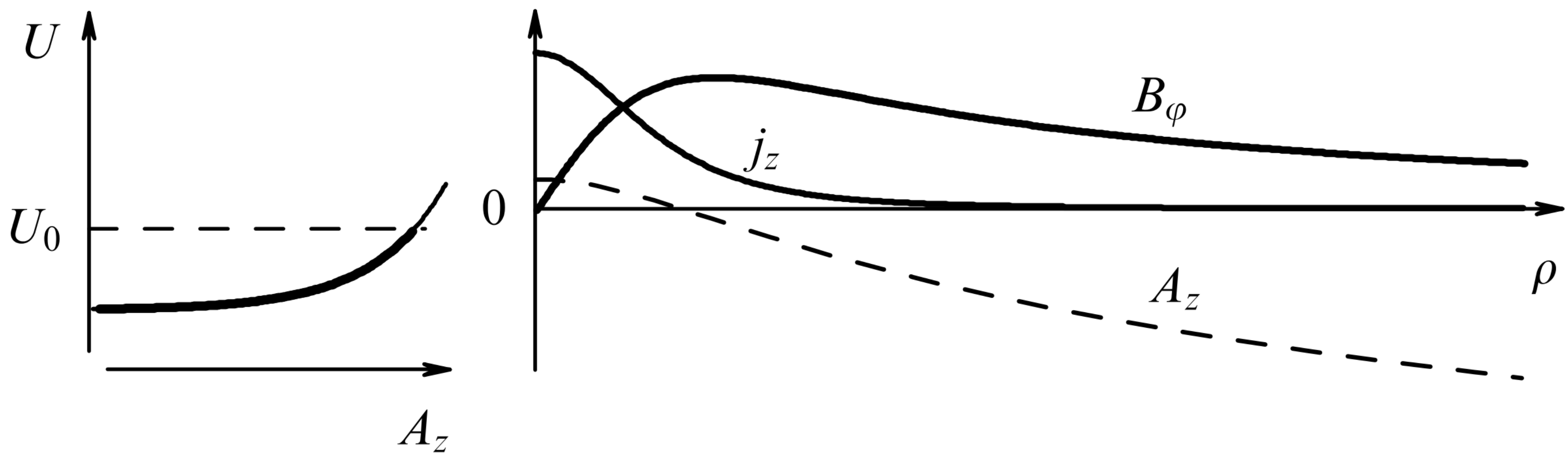


Figure 3. Typical profile of the potential of the Grad–Shafranov equation (71) and the dependences of A_z, B_φ, and j_z on the cylindrical coordinate ρ for the solutions given by formulas (72)–(74) in the form of the Bennett pinch. Hereinafter, the bold line in the Grad–Shafranov potential plots indicates the region of varying A_z values.

Examples of double-scale and tubular filaments

One more important class of current filaments is represented by solutions of the Grad–Shafranov equation for distribution functions in the form of the sum of two components that exhibit exponential dependence on the generalized momentum P_z with different exponents whose ratio equals w and may have arbitrary dependences on particle energy:

$$\frac{\mathrm{d}^2 A_z}{\mathrm{d}\rho^2} + \frac{1}{\rho}\frac{\mathrm{d}A_z}{\mathrm{d}\rho} = -\frac{W_1}{A_0}\exp\left(\frac{A_z}{A_0}\right) - \frac{W_2}{wA_0}\exp\left(\frac{A_z}{wA_0}\right). \quad (77)$$

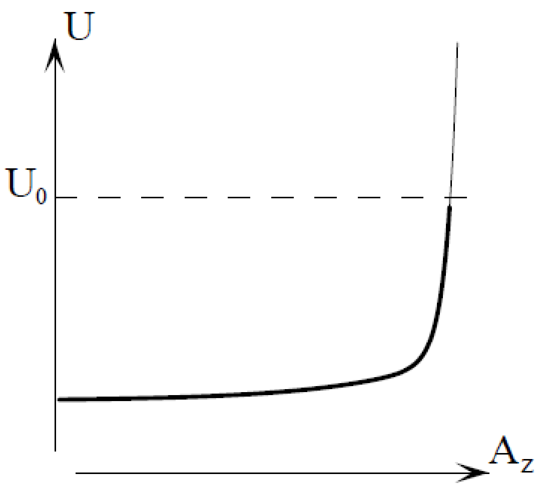
Here, as in the case of the generalized Bennett pinch, $A_0 = m_\alpha c^2/\zeta_\alpha e_\alpha$, and constants W_1 and W_2 are given by integrals of energy distributions. The most interesting case is that of exponents having identical signs, which allows us to consider A_0 and w, for certainty, as being positive. Moreover, it can be assumed, without a loss of generality, that $w < 1$, and one of the two exponents (second) changes faster than the other.

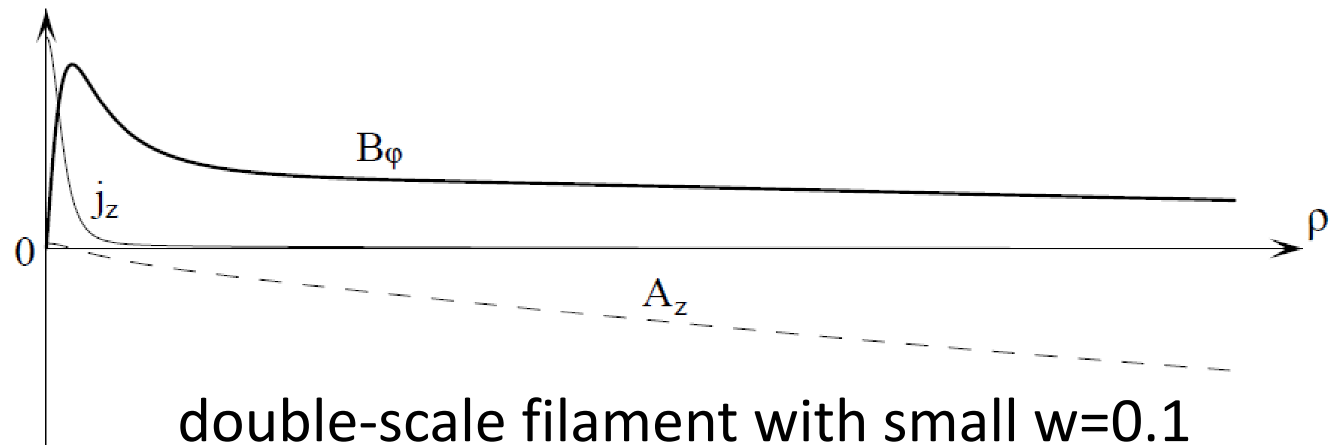
If $W_1 > 0$ and $W_2 > 0$, while $w \ll 1$, a double-scale solution is possible, as exemplified for $w = 0.1$ in Fig. 6a. If $W_1 > 0$, $W_2 < 0$, the solution will take the shape of a 'tubular' current filament in which a current density maximum is shifted from the cylindrical symmetry axis to a certain cylinder around it (Fig. 6b).

The total current in these two solutions differs from $2cA_0$ obtained upon direct generalization of the Bennett pinch, which accounts for the different asymptotic behavior of solutions far from the axis, despite the fact that only one of the two components on the right-hand side of Eqn (77) is essential at large ρ, and the form of the equation coincides with that of equation (71) having solutions in the form of Bennett type pinches. The analysis of the asymptotic behavior of such fragments at large ρ shows that the magnetic field decreases in inverse proportion to the distance from the axis.

Generalized relativistic Bennett pinch

$$\frac{d^2 A_y}{d\rho^2} + \frac{1}{\rho}\frac{dA_y}{d\rho} = -\frac{W_1}{A_0}\exp\left(\frac{A_y}{A_0}\right) - \frac{W_2}{wA_0}\exp\left(\frac{A_y}{wA_0}\right)$$





double-scale filament with small w=0.1

Limiting case of Bennett pinch with current along a wire on the axis I = (1-q)cA_0. If q > 1, the latter is opposite to the pinch current, which is expelled from the axis area and localized in a hollow tube.

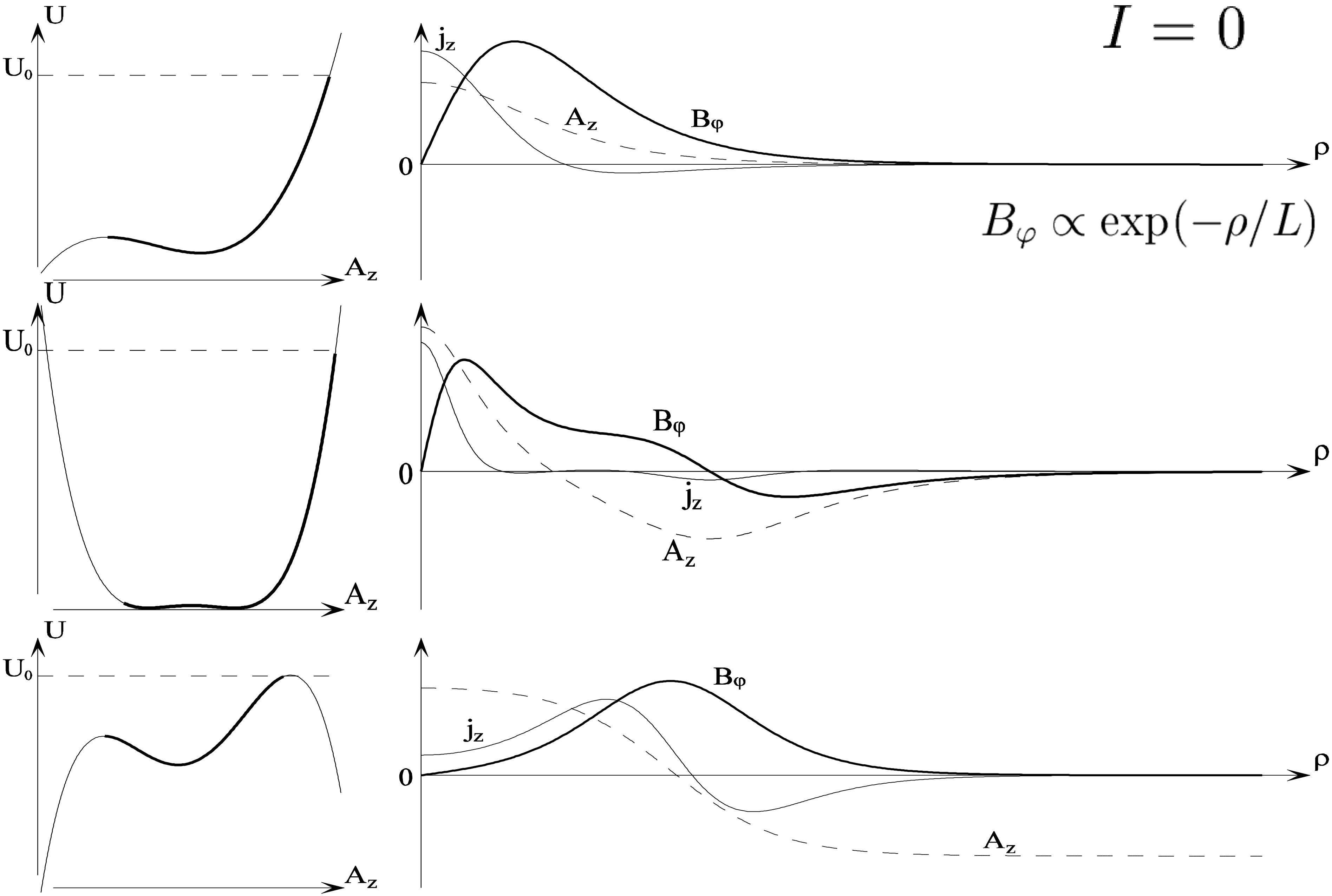
$$B_\varphi = \frac{2A_0\left(1-q+(1+q)(\kappa\rho)^{2q}\right)}{\rho(1+(\kappa\rho)^{2q})}$$

$$j_z = \frac{2cA_0 q^2\kappa^2}{\pi((\kappa\rho)^{1-q}+(\kappa\rho)^{1+q})^2}$$

Shielded current filaments (Taylor order d=3, d=4)

$$I = 0$$

$$B_\varphi \propto \exp(-\rho/L)$$



Other interesting classes of solutions emerge when using distribution functions of particles with a higher order of polynomial expansion in terms of the generalized momentum P_z. Then, the Grad–Shafranov potential as a function of the vector potential component A_z is a polynomial of the same order, and the spatial structure of the fragment can be rather complicated. For example, if the range of A_z variations is bounded and A_z monotonically, without oscillations, tends toward a certain constant, while ρ tends to infinity, the filament it describes is the central current surrounded by an equal and opposite current, such that the total current is zero (with more than one change in the current density sign being possible). Figure 5a presents an example of the solution of such a form for the Grad–Shafranov equation with the potential proportional to $A_z^3 - A_0 A_z^2$ at a certain constant A_0 value. This solution corresponds to such a set of PDF parameters at which the representative point of A_z as a variable of equation (70) tends to a local maximum in the Grad–Shafranov potential as $\rho \to \infty$. In this case, the only nonzero magnetic field component B_φ has one sign everywhere, and the current density changes the sign only once as the coordinate ρ grows.

If the Grad–Shafranov potential takes the form of a fourth-order polynomial, e.g., $U(A_z) \propto A_z^4 - A_0^2 A_z^2$ with a certain constant A_0, the solutions become possible with more than one change of current density direction and a few changes of the component B_φ sign. The solution with B_φ changing the sign only once and the current density being two cylindrical countercurrents embedded within each other is presented in Fig. 5b. The analysis of the asymptotic behavior of filaments of this type at large ρ reveals their exponential decrease.

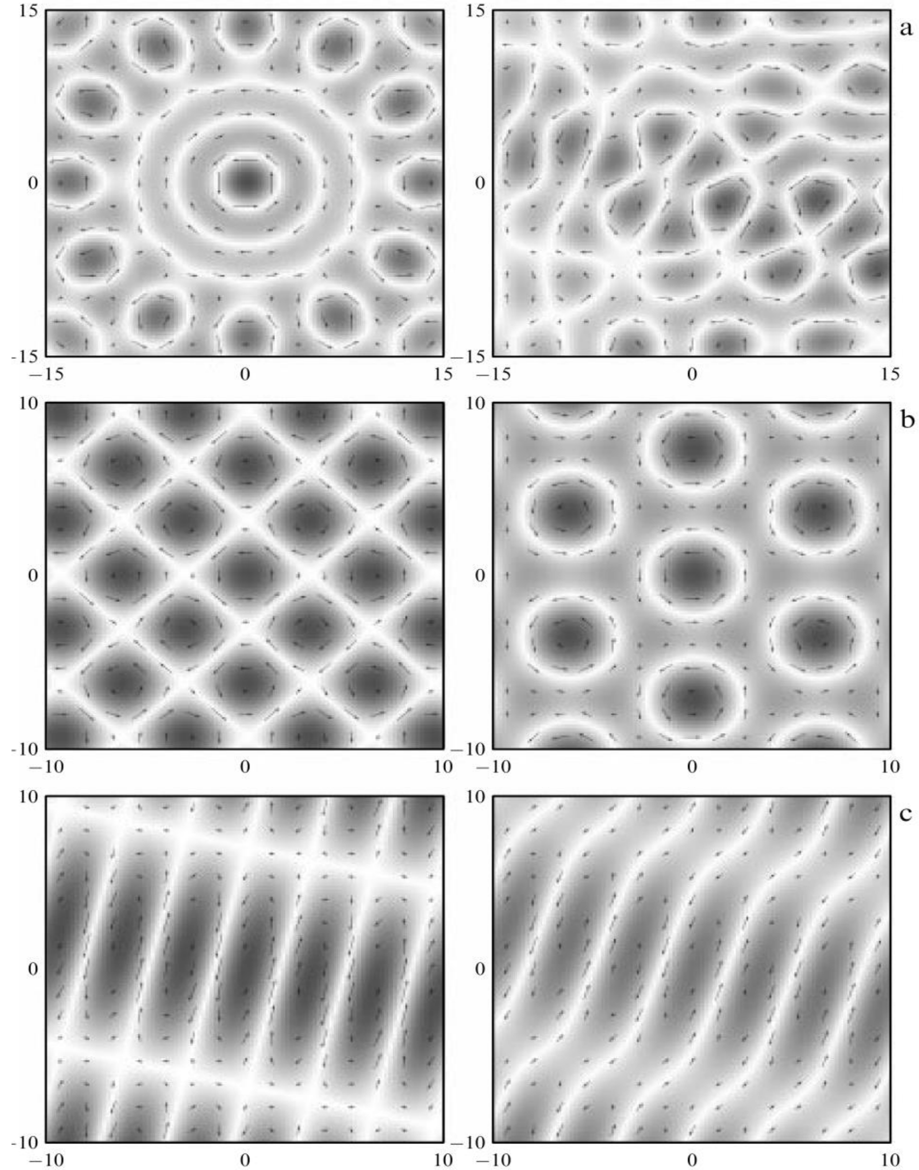


Figure 7. Two-dimensionally inhomogeneous current configurations. The shades of gray characterize the current density modulus (white color denotes zero density), and arrows indicate magnetic field magnitude and direction. The scale on each axis is counted in k^{-1} units.

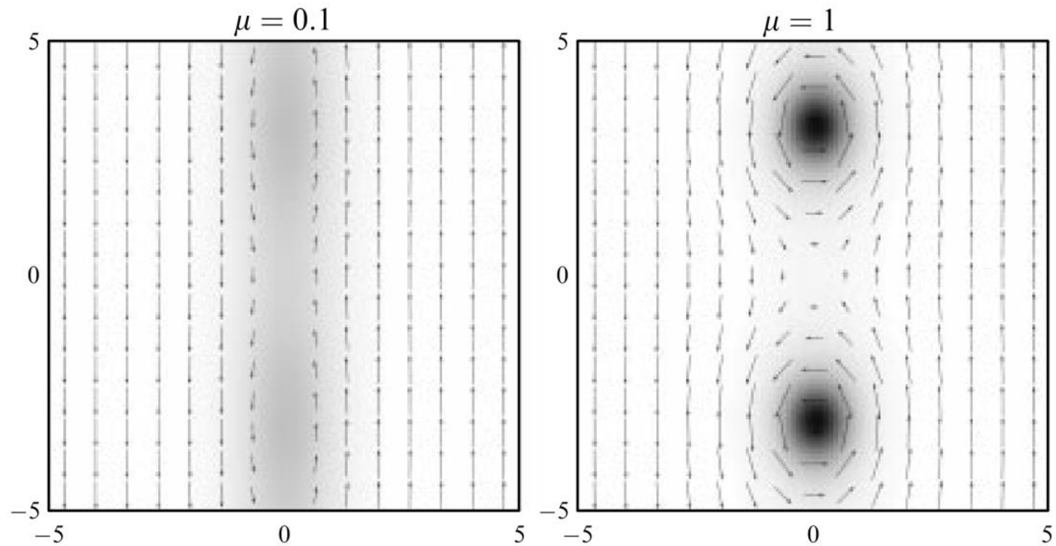


Figure 8. Two-dimensionally inhomogeneous configurations described by potential (81) at different values of the inhomogeneity parameter μ. The shades of gray show current density (white color denotes zero density), and arrows indicate magnetic field strength and direction. The scale on each axis is counted in κ^{-1} units.

Fadeev solution

$$A_z = -2A_0 \ln\left(\sqrt{1+\mu^2}\cosh(\kappa x) + \mu\cos(\kappa y)\right) + 2A_0 \ln\left(\kappa\sqrt{\frac{2A_0^2}{W_0}}\right). \tag{81}$$

Here, the free parameter μ is responsible for the degree of structural inhomogeneity along the y-coordinate. At $\mu = 0$, we obtain generalization of the Harris sheet for the case of arbitrary energy distribution of particles, which is discussed in detail in Section 3.2.1. Another free parameter κ^{-1} assigns the scale of magnetic field changes on the x- and y-axes. The equality of these parameters is essential.

The current density in solution (81) is given by the formula

$$j_z = -\frac{c}{4\pi}\,\Delta_{xy}A_z = \frac{cA_0\kappa^2}{2\pi\left(\sqrt{1+\mu^2}\cosh(\kappa x) + \mu\cos(\kappa y)\right)^2}, \tag{82}$$

indicating that a self-consistent current structure at large enough μ actually consists of a chain of generalized Bennett pinches arranged with a period of $2\pi/\kappa$ and carrying $2cA_0$ current each. The magnetic field and current density entering solution (81) at two different values of the imhomogeneity parameter μ are shown in Fig. 8.

Classification of the current filaments

For the cylindrically symmetrical filaments with a purely azimuthal magnetic field, the method of particle motion invariants admits only the following three qualitatively different types of the self-consistent structures.

I. The first type assumes an unlimited value of the vector potential A_z. This implies that the A_z is a monotonic function and the azimuthal component of the magnetic field is of the same sign for all values of the radial variable ρ. The current density may be sign-changing, although the current through any circular area perpendicular to z with a center on z has the same sign. A total current can be either finite or zero. The plasma can be localized near axis z, with its density exponentially vanishing with an increasing distance from the axis.

II. In the second type, the range of values of A_z is limited and its derivative with respect to ρ (the azimuthal component of magnetic field) changes sign a finite number of times. So, the “motion” of the oscillator (i.e., A_z) in the Grad-Shafranov potential starts at $\rho = 0$ with sliding down the slope of the well and ends on a local summit or, in a degenerate case, at the point where the first two derivatives of $U(A_z)$ with respect to A_z vanish.

A bottom of the potential well cannot be reached in an infinitely slow monotonic manner since a general solution to the oscillator equation with a viscous friction and a zero right-hand side is $A_z = c_1 + c_2 \cdot \ln \rho$. So, even for a completely flat bottom the motion is unlimited and the “friction” cannot stop the motion at a finite distance. Between the beginning and the end of the motion, there could be a few reflections from the potential walls that are higher than the final summit. The total current is absent, the magnetic field declines faster than $1/\rho$ with the increase of ρ. In the general case, the summit (hilltop) on the vector potential's profile has a quadratic form. Hence, the magnetic field and the current density decline exponentially.

III. In the third type, the range of values of A_z is limited, the A_z oscillates infinitely many times near the well's bottom with the increase of ρ. In a general case, as the amplitude of these oscillations decreases, the profile of the well's bottom can be approximated by a parabola, which yields a Bessel-type solution. The amplitudes of oscillations of A_z, the magnetic field and the current density decrease as $1/\rho$ when ρ increases.

In the case when the series expansion of the Grad-Shafranov potential near the bottom starts with a term higher than quadratic, the infinitely many oscillations also exist, although their period grows with ρ. The resulting current configuration constitutes a finite or infinite collection of concentric current cylinders, among which the ones on the outside have, as a rule, a larger value of the total current and a lower current density than the inside ones.

Conclusions

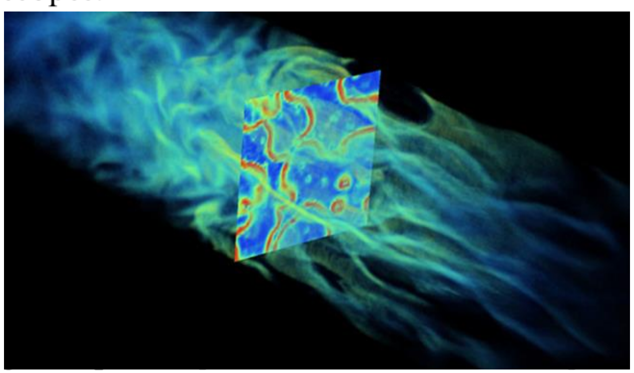
We investigate in detail several simplest cases of the exponential, sum of two exponentials, quadratic, third- and fourth-order polynomial PDFs.

In the first case we find a generalization of the Bennett pinch for arbitrary energy distribution of particles, which influences strongly the filament size as compared to the Maxwellian distribution for a given value of total current. In the third case we come to a current density profile with the radial oscillations described by the Bessel function with a spatial scale which also depends strongly on the energy distribution of particles.

In more complicated other cases we obtain well localized filaments with zero or finite total current and show that there are possible solutions with one or more changes of current density direction and a few changes of sign of the azimuthal magnetic field.

Our analytical results may be applied to the interpretation of the numerical simulations of the collisionless shock waves and to the analysis of various long-lived magnetic structures in the astrophysical plasmas (jets, winds, accretion disks) and laboratory laser plasmas, including description of the individual filaments in a quasi-magnetostatic turbulence.

Construction of the theory of self-consistent configurations of the magnetic field and currents in a collisionless plasma is one of the most important problem in basic plasma physics, affecting, among other things, realization of far-reaching experiments with the so-called high-energy density laser plasma based on modern superpower lasers and elucidation of intriguing astrophysical phenomena (quasars, microquasars, stellar and pulsar wind, cosmic gamma-ray bursts, etc.) and events in near-space (shock waves and current sheets in the Sun's and planets' magnetospheres) that are becoming increasingly more observable using unique space vehicles and telescopes.



diversity of PDFs and spatial magnetic field configurations associated with self-consistent current structures also appears to be natural in the astrophysical plasma, including active areas of the chromospheres and magnetic star coronas, current sheets formed by the stellar wind, or during contacts between magnetized plasma regions with different directions of the magnetic field in double systems with accretion (accretion disks and jets corresponding to neutron stars and black holes)

Problems awaiting further research:

(1) How representative are distribution functions depending only on particle motion invariants compared with a variety of distribution functions describing all stationary self-consistent current structures? How adequately do the first distributions represent qualitatively (physically) similar current structures described by more general particle distributions?

(2) What self-consistent current structures are most (or least) stable and what is the hierarchy of their instabilities? For which particle distribution is the structure with a given current density profile most stable?

(3) To what degree do the classes of stationary current structures extend (or contract) under the effect of a magnetic field imposed, for example, across the current sheet and/or boundary conditions, e.g., in prescribing the input and output particle flows at the borders of the current structure?

(4) Is there a possibility of quasi-adiabatic (slow) deformation of current structures without their appreciable destruction and, if yes, under which conditions does it occur? Is it possible to macroscopically describe any current structure deformation or their interaction with each other without a detailed analysis of PDF evolution?

(5) How do interparticle collisions, quasi-stationary electric fields, and higher-frequency electromagnetic fields influence self-consistent current structures? When does this influence result in their destruction or evolution of their macroscopic parameters without destruction, even if with a loss of energy content?