From Scatter-Free to Diffusive Propagation of Energetic Particles: Exact Solution of Fokker-Planck equation

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Overview

- Minimalist Model for CR (or SEP) transport: **Fokker-Planck Equation**
- Lacuna in Transport Description
- What we know for sure
  - ballistic propagation, $t \ll t_c(E)$
  - diffusive propagation, $t \gg t_c(E)$
- What is between the two limits and for how long?
  - "Telegraph" equation
  - hyper-diffusive corrections (Chapman-Enskog)
  - no specifics as to when to switch from $t \ll t_c$ to $t \gg t_c$
- **Exact Solution of Fokker-Planck Equation**
- Simplified Propagator for pitch-angle averaged FP solution
- Take Away
  - 2017PhRvD..95b3007M, arXiv:1703.02554

CR Transport Model: Fokker-Planck Equation

- CR transport driven by pitch- angle scattering, gyro-phase averaged

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial x} = \frac{\partial}{\partial \mu} \left(1 - \mu^2\right) D\left(E, \mu\right) \frac{\partial f}{\partial \mu}$$

- z -along $\mathbf{B}$; μ -cosine of CR pitch angle
- energy E enters as a parameter, but gain/loss terms $a\left(E\right)\partial f/\partial E$ can be removed by $E \to E' = \int a^{-1} dE - t$
- $D\left(\mu\right)$ is derived from a power index of the scattering turbulence, q
- for a power spectrum $P \propto k^{-q}$ (k is the wave number) $D\left(\mu\right) \propto \left|\mu\right|^{q-1}$
- more complex, anisotropic spectra, such as Goldreich-Shridhar 1995 $\to$ flat $D\left(\mu\right)$ except $\mu \approx 0, \pm 1$
- for review of yet more complex, fractional kinetics based transport see, e.g., Metzler & Klafter 2000, Zeleny & Milovanov 2004, Zaslavsky 2005 (book)
- **important case:** $q = 1 \quad \to \quad D = D\left(E\right)$

FP: $\partial_t f + v\mu\partial_x f = \partial_\mu \left(1-\mu^2\right) D\partial_\mu f$: diffusive approx.

- need evolution equation for

$$f_0\left(t,x\right) \equiv \left\langle f\left(t,x,\mu\right)\right\rangle \equiv \frac{1}{2}\int_{-1}^{1} f\left(\mu,t,x\right)d\mu.$$

- answer deems well known (e.g., Parker 65, Jokipii 66): average and expand in $1/D$:

$$\frac{\partial f_0}{\partial t} = -\frac{v}{2}\frac{\partial}{\partial x}\left\langle \left(1-\mu^2\right)\frac{\partial f}{\partial \mu}\right\rangle \quad \text{(exact eq.)}, \qquad \frac{\partial f}{\partial \mu} \simeq -\frac{v}{2D}\frac{\partial f_0}{\partial x}$$

- equation for f_0

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial x}\kappa\frac{\partial f_0}{\partial x}, \qquad \kappa = \frac{v^2}{4}\left\langle \frac{1-\mu^2}{D}\right\rangle = \frac{1}{6}\frac{v^2}{D\left(E\right)}$$

FP: $\partial_t f + v\mu\partial_x f = \partial_\mu \left(1-\mu^2\right) D\partial_\mu f$ diff.: limitation

- Critical step: $\partial f/\partial t$ is neglected compared to $v\partial f/\partial x$
- Justification: for $Dt \gtrsim 1$, $\tilde{f}(\mu) = f - f_0$ decays $\propto e^{-\lambda_1 Dt}$
- However, strong inhomogeneity $\to$ sharp anisotropy (real problem!)
- Cannot handle fundamental (Green's function) solution

Example

CR Transport Modeling

- $\kappa \sim v^2/D(E)$, galactic CR $\kappa \sim 10^{28} cm^2/s$, $\kappa \propto E^\alpha$, $\alpha \simeq 0.3 - 0.6$
- CR mfp $\lambda_{CR} \sim$ 1pc for a few 10 GeV particles
- Near the "knee" at $\simeq 3 \cdot 10^{15}$GeV, m.f.p. ~ 100 pc

Lacuna in CR Transport Model

- nearby sources of CRs are likely within this range of a few 100's pc
- cannot be studied within diffusive approach at the knee energy and beyond
- circumstantial evidence:
  - Sharp anisotropy in CR arrival directions, $\sim 10^{\circ}$ (Milagro data, *Abdo et al 2008*)
  - "nondiffusive transport" explanation: *MM, Diamond, Drury & Sagdeev 2010*

$$\partial_t f + v\mu\partial_x f = \partial_\mu \left(1-\mu^2\right) D\partial_\mu f$$

- approach this difficult part of parameter space (E) and CR propagation history from the other end: scatter-free regime: $t \ll 1/D\,(E)$

Fokker-Planck $\partial_t f + v\mu\partial_x f = \partial_\mu \left(1-\mu^2\right) D\partial_\mu f$

- discard collision term

$$\frac{\partial f}{\partial t} + v\mu\frac{\partial f}{\partial x} = 0$$

- solution

$$f\left(x,\mu,t\right) = f\left(x - v\mu t,\mu,0\right)$$

- consider a point source with initially isotropic distribution:

$$f\left(x,\mu,0\right) = \left(1/2\right)\delta\left(x\right)\Theta\left(1-\mu^2\right)$$

  δ and Θ - Dirac's delta and Heaviside unit step functions
- $\left\langle x^2\right\rangle = v^2t^2/3$: free escape with mean square velocity $v/\sqrt{3}$

$$\left\langle f\left(\mu,x,t\right)\right\rangle = f_0\left(x,t\right) = \left(2vt\right)^{-1}\Theta\left(1-x^2/v^2t^2\right)$$

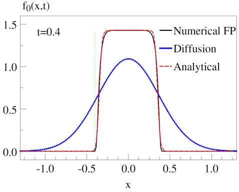
- expanding 'box' of decreasing height, $\propto 1/t$

Fokker-Planck $\partial_t f + v\mu\partial_x f = \partial_\mu \left(1-\mu^2\right) D\partial_\mu f$

- adopted $D(\mu) = const$ $(q=1)$ as both interesting and important case
- $\rightarrow$ *UNITS* : $D = v = 1$, $\left(Dt \rightarrow t, \quad \frac{D}{v}x \rightarrow x\right)$

$$\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} = \frac{\partial}{\partial \mu}\left(1-\mu^2\right)\frac{\partial f}{\partial \mu}$$

- contains no parameters: to correctly describe transition from ballistic to diffusive transport at times $t \sim 1$ $(\sim t_{col})$, we need **exact solution**



$$f = \begin{cases} (2t)^{-1}\,\Theta\left(1 - x^2/t^2\right), & t \ll t_c \\ \sqrt{\frac{3}{2\pi t}}e^{-3x^2/2t}, & t \gg t_c \end{cases}$$

FP: past/recent attempts at bridging the gap

$\partial_t f + v\mu\partial_x f = \partial_\mu \left(1-\mu^2\right) D\partial_\mu f \rightarrow$ Telegraph Equation

- In diff. derivation, retain $\partial f/\partial t$ in addition to $\partial f/\partial x$ corrections $\rightarrow \partial^2 f_0/\partial t^2$and higher derivative terms in p-a averaged equation, Axford 1965, Earl 1973++, Pauls, Burger & Bieber,1993, Schwadron & Gombosi, 1994, Litvinenko & Schlickeiser 2013...., Tautz+ 2016
- end up with and advocate Telegraph equation:

$$\frac{\partial f_0}{\partial t} - \frac{\partial}{\partial x}\kappa\frac{\partial f_0}{\partial x} + \tau\frac{\partial^2 f_0}{\partial t^2} = 0$$

  where $\tau \sim 1/D$, $\kappa \sim v^2/D$
- TE is inconsistent with Chapman-Enskog expansion
- does not conserve number of particles without adding singular, $\delta\left(x - Vt\right)$ components (non-existing in FP).... MM & Sagdeev 2015, MM 2015

Fokker-Planck $\partial_t f + v\mu\partial_x f = \partial_\mu \left(1-\mu^2\right) D\partial_\mu f$

Analytic solution, step by step (ridiculously simple)

1. normalize f to unity

$$\int_{-\infty}^{\infty} dx \int_{-1}^{1} f d\mu/2 = 1$$

2. organize the moments of f into the following matrix

$$M_{ij} = \left\langle \mu^i x^j \right\rangle = \int_{-\infty}^{\infty} dx \int_{-1}^{1} \mu^i x^j f d\mu/2$$

3. for any $i, j \geq 0$, multiplying FP eq. by $\mu^i x^j$ and integrating, obtain a matrix equation for the moments M_{ij}:

$$\frac{d}{dt} M_{ij} + i\left(i+1\right) M_{ij} = j M_{i+1,j-1} + i\left(i-1\right) M_{i-2,j}$$

$$\partial_t M_{ij} + i\,(i+1)\,M_{ij} = jM_{i+1,j-1} + i\,(i-1)\,M_{i-2,j}$$

- evolution equation for M_{ij} depends on a "higher" moment $M_{i+1,j-1}$
- needs closure or truncation? Follow the footsteps of hydrodynamics derivation?
- surprisingly, it does not!
- equation couples anti-diagonal elements from two closest nonadjacent anti-diagonals
- set of moments $M_{ij}\,(t)$ can be subsequently resolved to any order $n = i + j$
- Indeed, as $M_{00} = 1$, and $M_{ik} = M_{ki} = 0$ for any $i < 0$, $k \geq 0$

$$\partial_t M_{ij} + i\,(i+1)\,M_{ij} = jM_{i+1,j-1} + i\,(i-1)\,M_{i-2,j}$$

$$M = \begin{pmatrix} 1 & \langle x \rangle & \langle x^2 \rangle & \langle x^3 \rangle \\ \langle \mu \rangle & \langle \mu x \rangle & \langle \mu x^2 \rangle & \nearrow \\ \langle \mu^2 \rangle & \langle \mu^2 x \rangle & \nearrow & \iddots \\ \langle \mu^3 \rangle & \nearrow & \iddots & \\ \nearrow & \iddots & & \end{pmatrix}$$

- matrix elements can be subsequently found on each anti-diagonal working as shown by arrows
- first two moments on the uppermost antidiagonal are
- $M_{10}\,(t) = \langle \mu \rangle = \langle \mu \rangle_0 \exp\left(-2t\right)$ and
  $M_{01} = \langle x \rangle = \langle x \rangle_0 + \langle \mu \rangle_0 \left[1 - \exp\left(-2t\right)\right]/2$
- higher moments can be obtained inductively

General Solution for the moments

$$M_{ij}(t) = M_{ij}(0)\, e^{-i(i+1)t} + \int_0^t e^{i(i+1)(t'-t)}$$

$$\times \left[j M_{i+1,j-1}\left(t'\right) + i\,(i-1)\, M_{i-2,j}\left(t'\right) \right] dt'$$

- all higher moments can be obtained in form of series in $t^k e^{-nt}$, where k and n are integral numbers
- set of moments on the third anti-diagonal, M_{20}, M_{11}, M_{02} (known since 1922, G.I. Taylor):

$$M_{20} = \frac{1}{3}, \quad M_{11} = \frac{1}{6}\left(1 - e^{-2t}\right), \quad M_{02} = M_{02}(0) + \frac{t}{3} - \frac{1}{6}\left(1 - e^{-2t}\right)$$

- $M_{02} \equiv \left\langle x^2 \right\rangle \propto t^2$ ($t \ll 1$, ballistic propagation , $\left\langle x^2 \right\rangle \propto t$ (diffusive propagation)
- for simplicity, assume initial $f(x, \mu, 0)$ symmetric in x and μ
- this eliminates all odd moments at $t = 0$
- sufficient for the fundamental solution: $M_{02}(0) = \left\langle x^2 \right\rangle_0 = 0$

Higher moments and moment generating function

- however, just a few moments do not yield accurate solution
- critical to sum up infinite series, but they grow (!)

$$M_{08} = \frac{1}{6945750}e^{-20t} - \frac{5t+2}{253125}e^{-12t} +$$

$$\left(\frac{t^2}{567} + \frac{11t}{11907} - \frac{59}{27783}\right)e^{-6t} - \left(\frac{14}{25}t^3 + \frac{858}{125}t^2 + \frac{151042}{5625}t + \frac{18509371}{506250}\right)$$

$$\times e^{-2t} + \frac{35}{27}t^4 - \frac{224}{27}t^3 + \frac{3554}{135}t^2 - \frac{281183}{6075}t + \frac{123403}{3375}$$

- For any t, leading terms can be identified and summed up, using a general expression for moment generating function

$$f_\lambda(t) = \int_{-\infty}^{\infty} f_0(x,t)\, e^{\lambda x} dx = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} M_{0,2n}(t)$$

Summing up the moments

- need to sum for arbitrary λt (to capture sharp fronts). First, separately for $t < 1$

$$f_\lambda(t) = \frac{1}{\lambda t'}\sinh(\lambda t') + \frac{t^2}{45}\left[2\cosh(\lambda t) + \left(\lambda t - \frac{2}{\lambda t}\right)\sinh(\lambda t)\right]$$

where $t' = t - t^2/3 + \ldots$

- $t > 1$ - similar result, can be unified with $t < 1$ case
- after taking inverse Fourier transform

$$f_0(x,t) = \frac{1}{2\pi}\int e^{ikx} f_{-ik}(t)\, dk$$

$$f_0(x,t) \approx \frac{1}{4y}\left[\operatorname{erf}\left(\frac{x+y}{\Delta}\right) - \operatorname{erf}\left(\frac{x-y}{\Delta}\right)\right]$$

  - $t \ll 1$, fronts at, $\pm y$, $y \approx t$, thickness $\Delta \approx 2t^2/3\sqrt{5}$.
- After proceeding through the transdiffusive phase, $t \sim 1$
  - $y \approx (11t/6)^{1/4}$ and $\Delta \approx (2t/3)^{1/2}$ for $t \gg 1$

Universal Propagator $f_0(x,t) \approx \frac{1}{4y}\left[\operatorname{erf}\left(\frac{x+y}{\Delta}\right) - \operatorname{erf}\left(\frac{x-y}{\Delta}\right)\right]$

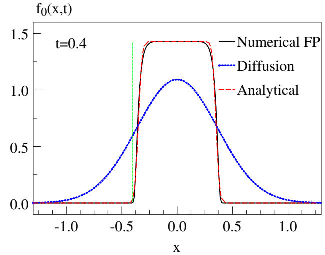
- the same form for all $0 < t < \infty$
- the only difference in $y(t)$, and $\Delta(t)$ for $t \ll 1$ and $t \gg 1$
- suggests determination of y and Δ from *exact* relations:

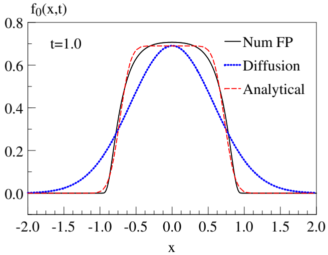
$$M_2 = \int x^2 f_0(x,t)\,dx, \quad M_4 = \int x^4 f_0(x,t)\,dx$$

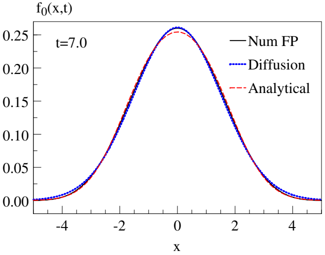
$$y = \left[\frac{45}{2}\left(M_2^2 - \frac{1}{3}M_4\right)\right]^{1/4}, \quad \Delta = \sqrt{2M_2 - \sqrt{10}\sqrt{M_2^2 - \frac{1}{3}M_4}}$$

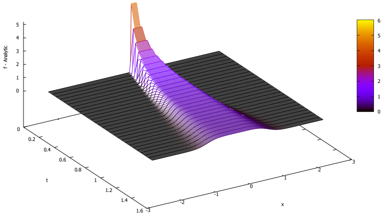
$$M_2 = \frac{t}{3} - \frac{1}{6}\left(1 - e^{-2t}\right), \quad M_4 = \frac{1}{270}e^{-6t} - \frac{t+2}{5}e^{-2t} + \frac{1}{3}t^2 - \frac{26}{45}t + \frac{107}{270}$$

Comparison with ballistic, diffusive, and numerical

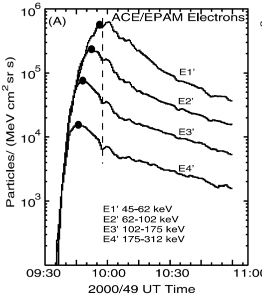


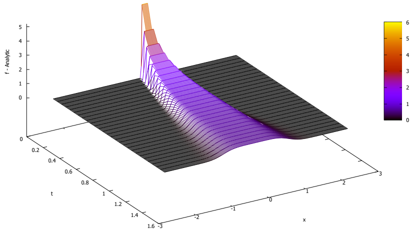


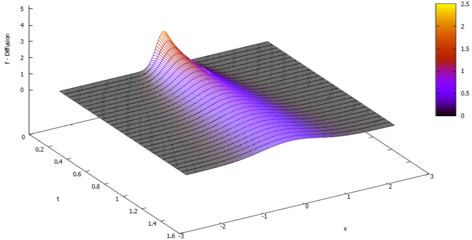




Preliminary qualitative comparison with observations







Haggerty and Roelof, 2002

Conclusions

- Fokker-Planck equation, commonly used for describing CR and other transport phenomena, is solved exactly
- The overall CR propagation can be categorized into three phases: ballistic ($t < 1$), transdiffusive ($t \sim 1$) and diffusive ($t \gg 1$), (time in units of collision time t_c).
- ballistic phase: source expands as a "box" of size $\Delta x \propto \sqrt{\langle x^2 \rangle} \propto t$ with "walls" at $x = \pm y(t) \approx \pm t$ of the width of each wall, $\Delta \propto t^2$.
- transdiffusive phase: box's walls thickened to the box size $\Delta \sim \Delta x \sim y$, slower expansion
- diffusion phase: $\Delta x \sim \Delta \propto \sqrt{t}$, the walls are completely smeared out, as $y \propto t^{1/4}$, so $y \ll \Delta$.
- the conventional diffusion approximation can be safely applied but, only after 5-7 collision times, depending on the accuracy requirements
- a popular telegraph approach, originally intended to cover also the earlier propagation phases at $t \lesssim 1$, is inconsistent with the exact FP solution
- no signatures of extended (sub) super-diffusive propagation regimes are present in the exact FP solution for $D(\mu) = const$