From Scatter-Free to Diffusive Propagation of Energetic Particles:
Exact Solution of Fokker-Planck equation

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Overview

- Minimalist Model for CR (or SEP) transport: Fokker-Planck Equation
- Lacuna in Transport Description
- What we know for sure
  - ballistic propagation, $t \ll t_c(E)$
  - diffusive propagation, $t \gg t_c(E)$
- What is between the two limits and for how long?
  - “Telegraph” equation
  - hyper-diffusive corrections (Chapman-Enskog)
  - no specifics as to when to switch from $t \ll t_c$ to $t \gg t_c$
- Exact Solution of Fokker-Planck Equation
- Simplified Propagator for pitch-angle averaged FP solution
- Take Away
CR Transport Model: Fokker-Planck Equation

- CR transport driven by pitch-angle scattering, gyro-phase averaged

\[
\frac{\partial f}{\partial t} + v_{\mu} \frac{\partial f}{\partial x} = \frac{\partial}{\partial \mu} (1 - \mu^2) D(E, \mu) \frac{\partial f}{\partial \mu}
\]

- \(z\)-along \(B\); \(\mu\)-cosine of CR pitch angle
- energy \(E\) enters as a parameter, but gain/loss terms \(a(E) \frac{\partial f}{\partial E}\) can be removed by \(E \rightarrow E' = \int a^{-1} dE - t\)
- \(D(\mu)\) is derived from a power index of the scattering turbulence, \(q\)
- for a power spectrum \(P \propto k^{-q}\) (\(k\) is the wave number) \(D(\mu) \propto |\mu|^{q-1}\)
- more complex, anisotropic spectra, such as Goldreich-Shridhar 1995 \(\rightarrow\) flat \(D(\mu)\) except \(\mu \approx 0, \pm 1\)
- **important case:** \(q = 1\) \(\rightarrow\) \(D = D(E)\)
FP: $\partial_t f + v \mu \partial_x f = \partial_\mu (1 - \mu^2) D \partial_\mu f$: diffusive approx.

- need evolution equation for

$$f_0(t, x) \equiv \langle f(t, x, \mu) \rangle \equiv \frac{1}{2} \int_{-1}^{1} f(\mu, t, x) d\mu.$$ 

- answer deems well known (e.g., Parker 65, Jokipii 66): average and expand in $1/D$:

$$\frac{\partial f_0}{\partial t} = -\frac{v}{2} \frac{\partial}{\partial x} \left\langle (1 - \mu^2) \frac{\partial f}{\partial \mu} \right\rangle \quad \text{(exact eq.)}, \quad \frac{\partial f}{\partial \mu} \simeq -\frac{v}{2D} \frac{\partial f_0}{\partial x}$$

- equation for $f_0$

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial x} \kappa \frac{\partial f_0}{\partial x}, \quad \kappa = \frac{v^2}{4} \left\langle \frac{1 - \mu^2}{D} \right\rangle = \frac{1}{6} \frac{v^2}{D (E)}$$
FP: $\partial_t f + v\mu \partial_x f = \partial_\mu (1 - \mu^2) D \partial_\mu f$ diff.: limitation

- Critical step: $\partial f / \partial t$ is neglected compared to $v \partial f / \partial x$
- Justification: for $Dt \gtrsim 1$, $\tilde{f}(\mu) = f - f_0$ decays $\propto e^{-\lambda_1 Dt}$
- However, strong inhomogeneity $\rightarrow$ sharp anisotropy (real problem!)
- Cannot handle fundamental (Green’s function) solution

Example

CR Transport Modeling

- $\kappa \sim v^2 / D(E)$, galactic CR $\kappa \sim 10^{28} \text{cm}^2/s$, $\kappa \propto E^\alpha$, $\alpha \simeq 0.3 - 0.6$
- CR mfp $\lambda_{CR} \sim 1\text{pc}$ for a few $10$ GeV particles
- Near the “knee” at $\simeq 3 \cdot 10^{15}\text{GeV}$, m.f.p. $\sim 100 \text{ pc}$
nearby sources of CRs are likely within this range of a few 100’s pc
cannot be studied within diffusive approach at the knee energy and beyond
circumstantial evidence:
- Sharp anisotropy in CR arrival directions, $\sim 10^\circ$ (Milagro data, Abdo et al 2008)
- “nondiffusive transport” explanation: MM, Diamond, Drury & Sagdeev 2010

$$\partial_t f + v\mu \partial_x f = \partial_\mu (1 - \mu^2) D \partial_\mu f$$

approach this difficult part of parameter space ($E$) and CR propagation history from the other end: scatter-free regime: $t \ll 1/D(E)$
Fokker-Planck \( \partial_t f + v \mu \partial_x f = \partial_\mu (1 - \mu^2) D \partial_\mu f \)

- discard collision term

\[
\frac{\partial f}{\partial t} + v \mu \frac{\partial f}{\partial x} = 0
\]

- solution

\[
f (x, \mu, t) = f (x - v \mu t, \mu, 0)
\]

- consider a point source with initially isotropic distribution:

\[
f (x, \mu, 0) = (1/2) \delta (x) \Theta (1 - \mu^2)
\]

\( \delta \) and \( \Theta \) - Dirac’s delta and Heaviside unit step functions

- \( \langle x^2 \rangle = v^2 t^2 / 3 \): free escape with mean square velocity \( v/\sqrt{3} \)

\[
\langle f(\mu, x, t) \rangle = f_0 (x, t) = (2vt)^{-1} \Theta (1 - x^2 / v^2 t^2)
\]

- expanding ‘box’ of decreasing height, \( \propto 1/t \)
Fokker-Planck \( \partial_t f + v \mu \partial_x f = \partial_\mu (1 - \mu^2) D \partial_\mu f \)

- adopted \( D(\mu) = \text{const} \ (q = 1) \) as both interesting and important case
- \( \rightarrow \text{UNITS} : D = v = 1, (Dt \rightarrow t, \frac{D}{v}x \rightarrow x) \)

\[
\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} = \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial f}{\partial \mu}
\]

- contains no parameters: to correctly describe transition from ballistic to diffusive transport at times \( t \sim 1 \ (\sim t_{col}) \), we need exact solution

\[
f = \begin{cases} 
(2t)^{-1} \Theta (1 - x^2/t^2), & t \ll t_c \\
\sqrt{\frac{3}{2\pi t}} e^{-3x^2/2t}, & t \gg t_c
\end{cases}
\]
FP: past/recent attempts at bridging the gap

\[ \partial_t f + \nu \mu \partial_x f = \partial_\mu (1 - \mu^2) D \partial_\mu f \rightarrow \text{Telegraph Equation} \]

- In diff. derivation, retain \( \partial f / \partial t \) in addition to \( \partial f / \partial x \) corrections → \( \partial^2 f_0 / \partial t^2 \) and higher derivative terms in p-a averaged equation, Axford 1965, Earl 1973++, Pauls, Burger & Bieber, 1993, Schwadron & Gombosi, 1994, Litvinenko & Schlickeiser 2013, ..., Tautz+ 2016

- end up with and advocate Telegraph equation:

\[ \frac{\partial f_0}{\partial t} - \frac{\partial}{\partial x} \kappa \frac{\partial f_0}{\partial x} + \tau \frac{\partial^2 f_0}{\partial t^2} = 0 \]

where \( \tau \sim 1/D, \kappa \sim \nu^2/D \)

- TE is inconsistent with Chapman-Enskog expansion
- does not conserve number of particles without adding singular, \( \delta (x - Vt) \) components (non-existing in FP).... MM & Sagdeev 2015, MM 2015
Fokker-Planck \( \partial_t f + v\mu \partial_x f = \partial_\mu (1 - \mu^2) D \partial_\mu f \)

**Analytic solution, step by step (ridiculously simple)**

1. normalize \( f \) to unity

\[
\int_{-\infty}^{\infty} dx \int_{-1}^{1} f d\mu/2 = 1
\]

2. organize the moments of \( f \) into the following matrix

\[
M_{ij} = \langle \mu^i x^j \rangle = \int_{-\infty}^{\infty} dx \int_{-1}^{1} \mu^i x^j f d\mu/2
\]

3. for any \( i, j \geq 0 \), multiplying FP eq. by \( \mu^i x^j \) and integrating, obtain a matrix equation for the moments \( M_{ij} \):

\[
\frac{d}{dt} M_{ij} + i (i + 1) M_{ij} = j M_{i+1,j-1} + i (i - 1) M_{i-2,j}
\]
\[
\partial_t M_{ij} + i (i + 1) M_{ij} = j M_{i+1,j-1} + i (i - 1) M_{i-2,j}
\]

- evolution equation for \( M_{ij} \) depends on a “higher” moment \( M_{i+1,j-1} \)
- needs closure or truncation? Follow the footsteps of hydrodynamics derivation?
- surprisingly, it does not!
- equation couples anti-diagonal elements from two closest nonadjacent anti-diagonals
- set of moments \( M_{ij}(t) \) can be subsequently resolved to any order \( n = i + j \)
- Indeed, as \( M_{00} = 1 \), and \( M_{ik} = M_{ki} = 0 \) for any \( i < 0, k \geq 0 \)
\[ \partial_t M_{ij} + i(i + 1) M_{ij} = jM_{i+1,j-1} + i(i - 1) M_{i-2,j} \]

\[ M = \begin{pmatrix}
\langle x \rangle & \langle x^2 \rangle & \langle x^3 \rangle \\
\langle \mu \rangle & \langle \mu x \rangle & \langle \mu x^2 \rangle \\
\langle \mu^2 \rangle & \langle \mu^2 x \rangle \\
\langle \mu^3 \rangle & \\
\end{pmatrix} \]

- Matrix elements can be subsequently found on each anti-diagonal working as shown by arrows.
- First two moments on the uppermost antidiagonal are
  \[ M_{10}(t) = \langle \mu \rangle = \langle \mu \rangle_0 \exp(-2t) \]
  \[ M_{01} = \langle x \rangle = \langle x \rangle_0 + \langle \mu \rangle_0 [1 - \exp(-2t)]/2 \]
- Higher moments can be obtained inductively.
General Solution for the moments

\[ M_{ij} (t) = M_{ij} (0) e^{-i(i+1)t} + \int_0^t e^{i(i+1)(t' - t)} \]

\[ \times \left[ jM_{i+1,j-1} (t') + i (i - 1) M_{i-2,j} (t') \right] \, dt' \]

- all higher moments can be obtained in form of series in \( t^k e^{-nt} \), where \( k \) and \( n \) are integral numbers

- set of moments on the third anti-diagonal, \( M_{20}, M_{11}, M_{02} \) (known since 1922, G.I. Taylor):

\[ M_{20} = \frac{1}{3}, \quad M_{11} = \frac{1}{6} (1 - e^{-2t}), \quad M_{02} = M_{02} (0) + \frac{t}{3} - \frac{1}{6} (1 - e^{-2t}) \]

- \( M_{02} \equiv \langle x^2 \rangle \propto t^2 \) (ballistic propagation), \( \langle x^2 \rangle \propto t \) (diffusive propagation)

- for simplicity, assume initial \( f(x, \mu, 0) \) symmetric in \( x \) and \( \mu \)

- this eliminates all odd moments at \( t = 0 \)

- sufficient for the fundamental solution: \( M_{02} (0) = \langle x^2 \rangle_0 = 0 \)
higher moments and moment generating function

- however, just a few moments do not yield accurate solution
- critical to sum up infinite series, but they grow (!)

\[ M_{08} = \frac{1}{6945750} e^{-20t} - \frac{5t + 2}{253125} e^{-12t} + \]

\[ \left( \frac{t^2}{567} + \frac{11t}{11907} - \frac{59}{27783} \right) e^{-6t} - \left( \frac{14}{25} t^3 + \frac{858}{125} t^2 + \frac{151042}{5625} t + \frac{18509371}{506250} \right) \]

\[ \times e^{-2t} + \frac{35}{27} t^4 - \frac{224}{27} t^3 + \frac{3554}{135} t^2 - \frac{281183}{6075} t + \frac{123403}{3375} \]

- For any \( t \), leading terms can be identified and summed up, using a general expression for moment generating function

\[ f_\lambda (t) = \int_{-\infty}^{\infty} f_0 (x, t) e^{\lambda x} dx = \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} M_{0,2n} (t) \]
Summing up the moments

- need to sum for arbitrary $\lambda t$ (to capture sharp fronts). First, separately for $t < 1$

$$f_\lambda(t) = \frac{1}{\lambda t'} \sinh(\lambda t') + \frac{t^2}{45} \left[ 2 \cosh(\lambda t) + \left( \lambda t - \frac{2}{\lambda t} \right) \sinh(\lambda t) \right]$$

where $t' = t - t^2/3 + ...$

- $t > 1$ - similar result, can be unified with $t < 1$ case
- after taking inverse Fourier transform

$$f_0(x, t) = \frac{1}{2\pi} \int e^{ikx} f_{-ik}(t) \, dk$$

$$f_0(x, t) \approx \frac{1}{4y} \left[ \text{erf} \left( \frac{x + y}{\Delta} \right) - \text{erf} \left( \frac{x - y}{\Delta} \right) \right]$$

- $t \ll 1$, fronts at, $\pm y$, $y \approx t$, thickness $\Delta \approx 2t^2/3\sqrt{5}$.
- After proceeding through the transdiffusive phase, $t \sim 1$
  - $y \approx (11t/6)^{1/4}$ and $\Delta \approx (2t/3)^{1/2}$ for $t \gg 1$
Universal Propagator $f_0(x, t) \approx \frac{1}{4y} \left[ \text{erf} \left( \frac{x+y}{\Delta} \right) - \text{erf} \left( \frac{x-y}{\Delta} \right) \right]$

- the same form for all $0 < t < \infty$
- the only difference in $y(t)$, and $\Delta(t)$ for $t \ll 1$ and $t \gg 1$
- suggests determination of $y$ and $\Delta$ from exact relations:

$$M_2 = \int x^2 f_0(x, t) \, dx, \quad M_4 = \int x^4 f_0(x, t) \, dx$$

$$y = \left[ \frac{45}{2} \left( M_2^2 - \frac{1}{3} M_4 \right) \right]^{1/4}, \quad \Delta = \sqrt{2M_2 - \sqrt{10}} \sqrt{M_2^2 - \frac{1}{3} M_4}$$

$$M_2 = \frac{t}{3} - \frac{1}{6} (1 - e^{-2t}) \quad M_4 = \frac{1}{270} e^{-6t} - \frac{t + 2}{5} e^{-2t} + \frac{1}{3} t^2 - \frac{26}{45} t + \frac{107}{270}$$
Comparison with ballistic, diffusive, and numerical
Preliminary qualitative comparison with observations

\[ \begin{align*}
\text{Particles/ (MeV cm}^2\text{s sr)}
\end{align*} \]

\[ \begin{align*}
E1' 45-62 \text{ keV} \\
E2' 62-102 \text{ keV} \\
E3' 102-175 \text{ keV} \\
E4' 175-312 \text{ keV}
\end{align*} \]

\[ \begin{align*}
\text{2000/49 UT Time}
\end{align*} \]

Haggerty and Roelof, 2002
Conclusions

- Fokker-Planck equation, commonly used for describing CR and other transport phenomena, is solved exactly.
- The overall CR propagation can be categorized into three phases: **ballistic** \((t < 1)\), **transdiffusive** \((t \sim 1)\) and **diffusive** \((t \gg 1)\), (time in units of collision time \(t_c\)).
- **Ballistic phase**: source expands as a “box” of size \(\Delta x \propto \sqrt{\langle x^2 \rangle} \propto t\) with “walls” at \(x = \pm y(t) \approx \pm t\) of the width of each wall, \(\Delta \propto t^2\).
- **Transdiffusive phase**: box’s walls thickened to the box size \(\Delta \sim \Delta x \sim y\), slower expansion.
- **Diffusion phase**: \(\Delta x \sim \Delta \propto \sqrt{t}\), the walls are completely smeared out, as \(y \propto t^{1/4}\), so \(y \ll \Delta\).
- The conventional diffusion approximation can be safely applied but, only after 5-7 collision times, depending on the accuracy requirements.
- A popular telegraph approach, originally intended to cover also the earlier propagation phases at \(t \lesssim 1\), is inconsistent with the exact FP solution.
- No signatures of extended (sub) super-diffusive propagation regimes are present in the exact FP solution for \(D(\mu) = \text{const}\).